Connected cuts

Master thesis

Author Tomáš Turek

Institute Charles University

Date 2025

Table of Contents

Notation

- graph $G = (V, E)$, V - vertices, $E \subseteq \binom{V}{2}$ - edges
- $\{u, v\}$ - single edge
- $n := |V|$
- $[\ell] := \{1, 2, \ldots, \ell\}$ and $[k, \ell] := \{k, k+1, \ldots, \ell\}$
- $\langle \ell, r \rangle := \{x \in \mathbb{R} \mid \ell \leq x \leq r\}$

Definition The partition of vertices V into $\ell \in \mathbb{N}$ parts is a set $\mathcal{V} = \{V_1, V_2, \ldots, V_\ell\}$ such that $\bigcup_{i=1}^{\ell} V_i = V$ and $\forall i \neq j \in [\ell] : V_i \cap V_j = \emptyset$.

Definition A graph is connected if and only if every pair of vertices has a path between them.

Notation

Definition An induced subgraph $G[S]$ on vertices $S \subseteq V$ is a graph $G' = (S, E')$, s.t. $\{u, v\} \in E' \iff u \in S \wedge v \in S \wedge \{u, v\} \in E$. We say that $S \subseteq V$ is connected if the induced subgraph $G[S]$ is connected.

- $E(S, V \setminus S)$ - edges between S and $V \setminus S$
- $e(S, V \setminus S) = |E(S, V \setminus S)|$
- partition $S \subset V$ and $V \setminus S$ is a cut
- we also call $E(S, V \setminus S)$ a cut (the terms are interchangeable)

Minimum connected k-cut

Definition For a connected graph $G = (V, E)$ and $k \in \mathbb{N}$, a connected k-cut is $S \subset V$ such that all following properties hold:

1. $G[S]$ is connected.
2. $|S| = k$.

Problem *Minimum connected k-cut is to find connected k-cut minimizing $e(S, V \setminus S)$.*

Minimum connected k-cut with source

Definition For a connected graph $G = (V, E)$, a vertex $s \in V$ and $k \in \mathbb{N}$ a connected k-cut with source is $S \subset V$ such that all following properties hold:

1. $G[S]$ is connected.
2. $|S| = k$.
3. $s \in S$.

Problem *Minimum connected k-cut with source is to find connected k-cut with source minimizing $e(S, V \setminus S)$.*

MINIMUM CONNECTED k-CUT

Example

See a MINIMUM CONNECTED k-CUT example. For this graph we consider $k = 3$.

1. $G[S]$ is connected.
2. $|S| = k$.
3. $e(S, V \setminus S)$ is minimized.

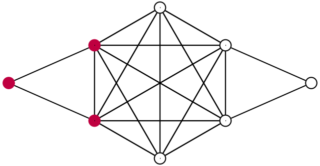


Figure: Possible solution with purple vertices chosen into S.

Question

What if we omit one of them.

Connected k-cut

Example

For this graph we consider $k = 3$.

1. $G[S]$ is connected.
2. $|S| = k$.
3. $e(S, V \setminus S)$ is minimized.

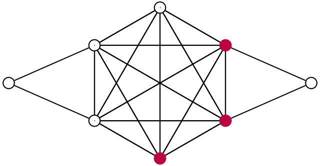


Figure: Possible solution with purple vertices chosen into S.

Solution

We may use only search algorithms.

Minimum connected cut

Example

For this graph we consider $k = 3$.

1. $G[S]$ is connected.
2. $|S| = k$.
3. $e(S, V \setminus S)$ is minimized.

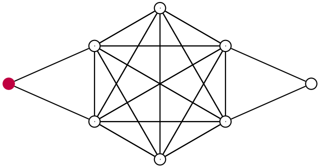


Figure: Possible solution with purple vertices chosen into S.

Solution

Either use flows and cuts or polyhedron by Garg [4].

Minimum k-cut

Example

For this graph we consider $k = 3$.

1. $G[S]$ is connected.
2. $|S| = k$.
3. $e(S, V \setminus S)$ is minimized.

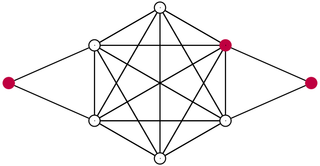


Figure: Possible solution with purple vertices chosen into S.

Solution

Use the results from minimum bisection by Feige and Krauthgamer [1].

Table of Contents

NP-completeness

- We use polynomial time reduction from MINIMUM BISECTION.
- We may assume that $|V|$ is even.
- The MINIMUM BISECTION problem is NP-complete [2, 3, 5].

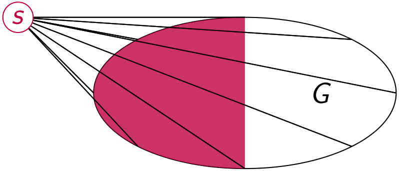
Problem *Given graph $G = (V, E)$ the goal is to find a partition $\mathcal{V} = \{V_1, V_2\}$ where $|V_1| = |V_2| = \frac{1}{2}|V|$ and $e(V_1, V_2)$ is minimized.*

NP-completeness of Minimum connected k-cut with source

Lemma *The* Minimum connected k-cut with source *problem is* NP*-complete.*

Proof.

- add vertex s
- set $k = \frac{n}{2} + 1$



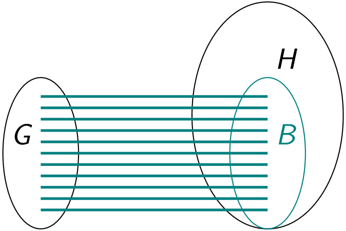
□

NP-completeness of MINIMUM CONNECTED k-CUT

Theorem *The* MINIMUM CONNECTED k-CUT *problem is* NP-*complete.*

Proof.

- create graph $H := K_{n^2}$ and choose n vertices into B
- connect B to G via matching and set $k = n^2 + \frac{n}{2}$



□

Table of Contents

Absorptive flow

Definition Given a graph $G = (V, E)$, $s \in V$ and an integer $k \in \mathbb{N}, k \leq n$, an absorptive flow is a pair of functions (f_V, f_E), $f_V : V \to \mathbb{R}$, $f_E : E \to \mathbb{R}$, satisfying the following properties:

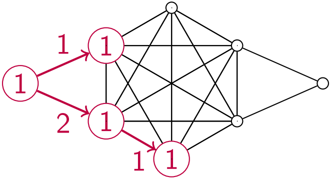
1. $\sum_{v \in V} f_V(v) = k$
2. $f_V(s) = 1$
3. $\forall v \in V : 0 \leq f_V(v) \leq 1$
4. $\sum_{v \in V, \{s,v\} \in E} f_E(\{s, v\}) = k - 1$
5. $\forall e \in E : 0 \leq f_E(e)$
6. $\forall v \in V \setminus \{s\} : \sum_{u \in V, \{u,v\} \in E} f_E(\{u, v\}) = \sum_{u \in V, \{v,u\} \in E} f_E(\{v, u\}) + f_V(v)$

Definition Given an absorptive flow (f_V, f_E) on a graph $G = (V, E)$ we set $S = \{v \in V \mid f_V(v) > 0\}$. The boundary of absorptive flow is $E(S, V \setminus S)$, where its size is $e(S, V \setminus S)$.

Examples of absorptive flows

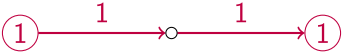
Example

Here we have an example of an absorptive flow for a given graph and $k = 4$.



Example

But the flow itself does not have to be connected. Lets see this example with $k = 2$.



Integer linear program (ILP) for an absorptive flow with min boundary

Variables

$$\forall v \in V \quad f_v = \begin{cases} 1 & \text{if the vertex } v \text{ absorbs.} \\ 0 & \text{otherwise.} \end{cases}$$

$\forall \{u, v\} \in E \quad f_{uv}, f_{vu} \geq 0$ the amount of flow on edge $\{u, v\}$ going in the respective direction.

$$\forall e \in E \quad x_e = \begin{cases} 1 & \text{if } e \in E(S, V \setminus S). \\ 0 & \text{otherwise.} \end{cases}$$

Objective

$$\min \sum_{e \in E} x_e$$

Integer linear program (ILP) for an absorptive flow with min boundary

Constraints

$$x_{\{u,v\}} \geq f_u - f_v \quad \forall \{u,v\} \in E$$

$$x_{\{u,v\}} \geq f_v - f_u \quad \forall \{u,v\} \in E$$

$$\sum_{v \in V, \{s,v\} \in E} f_{sv} = k - 1$$

$$f_s = 1$$

$$\sum_{u \in V, \{u,v\} \in E} f_{uv} = \sum_{u \in V, \{v,u\} \in E} f_{vu} + f_v \quad \forall v \in V \setminus \{s\}$$

$$\sum_{u \in V} f_u = k$$

$$(k - d(s,v)) \cdot f_v \geq \sum_{u \in V, \{uv\} \in E} f_{uv} \forall v \in V \setminus \{s\}$$

ILP properties

Lemma *The optimal solution* (x^*, f^*) *of ILP correspond to a* MINIMUM CONNECTED k-CUT *and vice versa.*

Lemma *The integrality gap between the integer linear program and its linear relaxation is at least* k.

Bound a relaxation of the ILP

- d_x is a pseudo-metric on $V \times V \to \mathbb{R}_0^+$; derived from x_e as a shortest path
- $\mathrm{OPT}_{\mathrm{LP}}$ denotes the optimum of linear program relaxation

Definition For a given graph $G = (V, E)$ and a pseudo-metric $d : V \times V \to \mathbb{R}_0^+$, a ball around vertex $u \in V$ with radius $0 \leq r \leq 1$ is a set $\{v \in V \mid d(u, v) \leq r\}$, denoted as $\mathcal{B}_d(u, r)$.

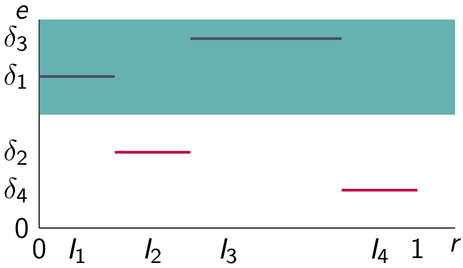
Lemma *For all vertices $v \in V$ the inequality $f_v \geq 1 - d_x(s, v)$ holds.*

Lemma *For any $0 \leq r < 1$ these bounds $1 \leq |\mathcal{B}_{d_x}(s, r)| \leq 1 + \frac{k-1}{1-r}$ holds.*

Bounds

Lemma *Given a constant $0 < c < 1$, let $\mathcal{I}_c = \{r \in \langle 0, 1\rangle \mid e(\mathcal{B}_{d_x}(s, r), V \setminus \mathcal{B}_{d_x}(s, r)) > \frac{1}{c}\mathrm{OPT}_{\mathrm{LP}}\}$. Then $\mathcal{I}_c$ corresponds to a set of subintervals of $\langle 0, 1\rangle$ and the sum of their length is at most c.*

Proof. Use the fact: $\sum_{i=1}^{\ell} \delta_i \, |I_i| = \mathrm{OPT}_{\mathrm{LP}}$.



□

Algorithm

- we propose a bicriteria approximation algorithm for the MINIMUM CONNECTED k-CUT WITH SOURCE problem, and consequently also for MINIMUM CONNECTED k-CUT problem
- the technique we use is known as a *Ball growing technique* [4]

Theorem *For a given graph $G = (V, E)$, $k \in \mathbb{N}$ and $0 < c < 1$ there exists a polynomial time algorithm, which produces a connected cut S, such that $1 \leq |S| \leq \mathcal{O}(k)$ and $e(S, V \setminus S) \leq \frac{1}{c}\mathrm{OPT}_{\mathrm{LP}}$.*

- proof follows from previous Lemmata and the fact, that the function is piece-wise constant
- the stricter c we choose the worse $|S|$ we may get, but there is no guarantee
- the choice of c does not change the time complexity

Summary

- defining MINIMUM CONNECTED k-CUT and MINIMUM CONNECTED k-CUT WITH SOURCE problems
- proving NP-completeness
- bicriteria approximation of LP relaxation with the usage of a growing ball technique

Thank you for your attention.

Bibliography

FEIGE, U.; KRAUTHGAMER, R. A Polylogarithmic Approximation of the Minimum Bisection. *SIAM Journal on Computing*. 2002, vol. 31, no. 4, pp. 1090–1118.

GAREY, M.; JOHNSON, D.; STOCKMEYER, L. Some simplified NP-complete graph problems. *Theoretical Computer Science*. 1976, vol. 1, no. 3, pp. 237–267. ISSN 0304-3975.

GAREY, M. R.; JOHNSON, D. S. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, CA, 1979.

GARG, N.; VAZIRANI, V. Multicommodity Flows and Approximation Algorithms. 1996, pp. 10–17.

KRAUTHGAMER, R. Minimum Bisection. In: KAO, M.-Y. (ed.). *Encyclopedia of Algorithms*. New York, NY: Springer New York, 2016, pp. 1294–1297. ISBN 978-1-4939-2864-4.